NEW HIGHER-ORDER STATISTICS BASED CRITERIA FOR THE DESIGN OF LINEAR PREDICTION ERROR FILTERS

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ABSTRACT

In this paper, we propose two criteria for the design of (minimum-phase) linear prediction error (LPE) filters with a set of non-Gaussian measurements $x(n)$. The first one requires a slice of Mth-order ($M \geq 3$) cumulants of $x(n)$ and the other requires a slice of third-order cumulants of the prediction error of $x(n)$. We theoretically show that when $x(n)$ is contaminated by additive Gaussian noise, the designed LPE filters based on the proposed criteria are identical to the conventional correlation-based LPE filter associated with the case that $x(n)$ is noise-free. Moreover, as the conventional LPE filter, coefficients of the cumulant-based LPE filter associated with the first criterion can be obtained by solving a set of symmetric Toeplitz linear equations. Finally, some simulation results are provided to support the analytical results.

1. INTRODUCTION

Linear prediction error (LPE) filters [1-4] have been widely used in various signal processing areas such as speech processing, seismic deconvolution and spectral estimation. Coefficients of conventional correlation-based LPE filters can be obtained by solving a set of symmetric Toeplitz linear equations (the well-known Yule-Walker equations) formed of correlations $r_{xx}(k)$ of the stationary signal $x(n)$ of interest. Therefore, it is sensitive to measurement noise because $r_{xx}(k)$ includes correlations of noise. Recently, Chi *et al.* [5-7] proposed some higher-order statistics (HOS) (known as cumulants) based criteria for the design of LPE filters when $x(n)$ is non-Gaussian. The designed HOS based LPE filters are inherently immune from additive Gaussian noise because all higher-order (≥ 3) cumulants of Gaussian processes are totally zero. In this paper, we further propose two new criteria using a single slice of higher-order cumulants. As the criteria reported in [5-7], one of the proposed criteria resorts to numerical optimization for obtaining the desired LPE filter, whereas coefficients of the desired LPE filter associated with the other criterion can be obtained by solving a set of symmetric Toeplitz linear equations. Some simulation results are provided to support the proposed criteria. Finally, we draw some conclusions.

2. NEW HOS BASED CRITERIA FOR THE DESIGN OF LPE FILTERS

Assume that $x(n)$, $n = 0, 1, ..., N-1$ are the given real stationary non-Gaussian noisy measurements based on the following convolution model

$$x(n) = y(n) + w(n) = u(n) * h(n) + w(n) \quad (1)$$

where $y(n) = u(n) * h(n)$ (noise-free measurements), $u(n)$ is a real, zero-mean, independent identically distributed (i.i.d) non-Gaussian process with variance σ_u^2 and Mth-order cumulant γ_M ($M \geq 3$), $w(n)$ is zero-mean Gaussian noise and $h(n)$ is the impulse response of a linear time-invariant (LTI) causal stable system. For ease of later use, let $Cum^{(M)}(x_1, x_2, \ldots, x_M)$ denote the Mth-order cumulant of random variables $\{x_1, x_2, \ldots, x_M\}$, $C_{M,x}(k_1, k_2, \ldots, k_M)$ denote the Mth-order cumulant function of $x(n)$, $\widehat{C}_{M,x}(k_1, k_2, \ldots, k_M)$ denote the Mth-order biased sample cumulant function of $x(n)$ and

$$F_M = \sum_{n=-\infty}^{\infty} h^M(n). \quad (2)$$

Let $v_p(n)$ be a pth-order causal FIR filter with $v_p(0) = 1$ and input $x(n)$. Then the output $e(n)$ (prediction error of $x(n)$) of the filter is given by

$$e(n) = x(n) * v_p(n) = x(n) + \sum_{i=1}^{p} v_p(i)x(n-i). \quad (3)$$

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The conventional pth-order LPE filter is the $v_p(n)$ such that the mean square error $E[e^2(n)]$ is minimum. The new HOS based criteria for the design of LPE filters are described in Theorem 1 as follows:

Theorem 1. Let $\widehat{v}_p(n)$ be the optimum $v_p(n)$ based on any of the following two criteria

$$\widetilde{J}(v_p(n)) = \frac{\left\{\sum_{k=-\infty}^{\infty} C_{3,e}(0,k)\right\}^2}{|V_p(z=1)|^2} \geq \widetilde{J}(\widehat{v}_p(n)) \quad (4)$$

$$J_M(v_p(n)) = \left\{ \sum_{k=-\infty}^{\infty} Cum^{(M)}(x(n),\ldots,x(n),e(n+k),e(n+k)) \right\}^2 \geq J_M(\widehat{v}_p(n)), \quad M \geq 3 \quad (5)$$

where $V_p(z)$ is the z-transform of $v_p(n)$. Then the $\widehat{v}_p(n)$ associated with $\widetilde{J}$ and the one associated with J_M are identical to the conventional pth-order LPE filter associated with $y(n)$ (noise-free measurements), as long as $\gamma_3 F_1 \neq 0$ for the former and $\gamma_M F_{M-2} \neq 0$ for the latter.

Proof: Let

$$\xi(n) = y(n) * v_p(n) = u(n) * g(n) \quad (6)$$

where

$$g(n) = h(n) * v_p(n). \quad (7)$$

Let us simplify the numerator of $\widetilde{J}$ as follows:

$$\sum_{k=-\infty}^{\infty} C_{3,e}(0,k) = \sum_{k=-\infty}^{\infty} C_{3,\xi}(0,k)$$

$$= \gamma_3 \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} g^2(n) g(n+k) \right\}$$

$$= \gamma_3 \left\{ \sum_{n=-\infty}^{\infty} g^2(n) \right\} \left\{ \sum_{n=-\infty}^{\infty} g(n) \right\}$$

$$= \frac{\gamma_3}{\sigma_u^2} E[\xi^2(n)] H(z=1) V_p(z=1) \quad (8)$$

Substituting (8) into $\widetilde{J}$ given by (4) yields

$$\widetilde{J} = \left\{ \frac{\gamma_3}{\sigma_u^2} \sum_{n=-\infty}^{\infty} h(n) \right\}^2 \left\{ E[\xi^2(n)] \right\}^2 = \rho_p^2(3) \quad (9)$$

where

$$\rho_p(M) = \left[\frac{\gamma_M F_{M-2}}{\sigma_u^2} \right] E[\xi^2(n)]. \quad (10)$$

Moreover, J_M can be simplified as follows

$$J_M = \left[\gamma_M \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h^{M-2}(n) g^2(n+k) \right]^2$$

$$= \left[\frac{\gamma_M}{\sigma_u^2} \left\{ \sum_{n=-\infty}^{\infty} h^{M-2}(n) \right\} \right]^2 \left[\sum_{n=-\infty}^{\infty} \sigma_u^2 g^2(n) \right]^2$$

$$= \left[\frac{\gamma_M F_{M-2}}{\sigma_u^2} \right]^2 \left\{ E[\xi^2(n)] \right\}^2 = \rho_p^2(M) \quad (11)$$

One can see, from (9) and (11), that minimizing $\widetilde{J}$ and J_M are equivalent to minimizing $E[\xi^2(n)]$ when $\gamma_3 F_1 \neq 0$ for the former and $\gamma_M F_{M-2} \neq 0$ for the latter. Thus we have completed the proof.

Moreover, Theorem 1 also implies the following fact.

(F1) Assume that $H(z)$ and $H_{MP}(z)$ are spectrally equivalent and $H_{MP}(z)$ is minimum-phase with $H_{MP}(z=\infty) = 1$. The optimum $\widehat{v}_p(n)$ (associated with either $\widetilde{J}$ or J_M) $\rightarrow \widehat{v}_p{}'(n)$ as $p \rightarrow \infty$ where $\widehat{v}_p{}'(n)$ is the impulse response of the inverse filter $1/H_{MP}(z)$. Furthermore, $e(n)$ can be viewed as the noisy output of an allpass system $H(z)/H_{MP}(z)$ (a phase distortion system) driven by the input $u(n)$.

Next, we present how to obtain the optimum LPE filters based on the proposed criteria. First of all, consider the criterion $\widetilde{J}$ given by (4) which is obviously a highly nonlinear function of the filter coefficients. Gradient type numerical algorithms can be used to search for the desired $\widehat{v}_p(n)$ by minimizing

$$\widetilde{J} = \frac{\left\{\sum_{k=-K}^{K} \widehat{C}_{3,e}(0,k)\right\}^2}{|V_p(z=1)|^2} \quad (12)$$

where the integer K must be chosen large enough such that $\sum_{k=-K}^{K} \widehat{C}_{3,e}\ (0,k) \approx \sum_{k=-\infty}^{\infty} \widehat{C}_{3,e}(0,k)$.

It can be easily shown that the other criterion J_M given by (5) can be expressed as

$$J_M = \left\{ \sum_{i=0}^{p} \sum_{j=0}^{p} v_p(i) v_p(j) c(j-i) \right\}^2 \quad (13)$$

where

$$c(i) = \sum_{k=-\infty}^{\infty} C_{M,x}(0,\ldots,0,k,k+i) = c(-i). \quad (14)$$

Setting the partial derivative of J_M with respect to $v_p(i)$, $i = 1, 2, \ldots, p$ equal to zero, one can obtain the following linear equations

$$\sum_{j=0}^{p} \widehat{v}_p(j) c(j-i) = 0, \;\; i = 1, 2, \ldots, p \tag{15}$$

which can be rewritten in a matrix equation as

$$C\,\widehat{\mathbf{v}} = -\mathbf{c} \tag{16}$$

where $\widehat{\mathbf{v}} = (\widehat{v}_p(1), \widehat{v}_p(2), \ldots, \widehat{v}_p(p))^T$ is a $p \times 1$ vector, C is a $p \times p$ symmetric Toeplitz matrix with the (i,j)th component given by

$$C_{i,j} = c(i-j), \;\; 1 \le i \le p, \;\; 1 \le j \le p \tag{17}$$

and $\mathbf{c}$ is a $p \times 1$ vector given by

$$\mathbf{c} = (c(1), c(2), \ldots, c(p))^T. \tag{18}$$

Furthermore, it can be shown that the discrete-time Fourier transform $C(f)$ of $c(i)$ is given by

$$C(f) = |H(f)|^2 \gamma_M F_{M-2} \tag{19}$$

where $H(f) = H(z = exp\{j2\pi f\})$. The following fact can be easily inferred from (19).

(F2) The sequence $c(i) = c(-i)$ is positive definite if $\gamma_M F_{M-2} > 0$, and negative definite if $\gamma_M F_{M-2} < 0$. The desired $\widehat{v}_p(n)$ obtained by solving (16) is minimum-phase.

The well-known computationally efficient Levison-Durbin recursive algorithm [1,2] can be used to solve (16) for $\widehat{\mathbf{v}}$. It is summarized as follows:

For $l = 1$,

$$v_1(1) = -\frac{c(1)}{c(0)} \tag{20}$$

$$\rho_1(M) = (1 - v_1^2(1)) c(0) \tag{21}$$

For $2 \le l \le p$,

$$v_l(l) = -\frac{1}{\rho_{l-1}(M)} \left\{ c(l) + \sum_{j=1}^{l-1} v_{l-1}(j) c(l-j) \right\} \tag{22}$$

$$v_l(i) = v_{l-1}(i) + v_l(l) v_{l-1}(l-i), \quad i = 1, 2, \cdots, l-1 \tag{23}$$

$$\rho_l(M) = (1 - v_l^2(l)) \rho_{l-1}(M) \tag{24}$$

Note, from (11), that $|\rho_l(M)| = J_M^{1/2}(\widehat{v}_l(n)) \le J_M^{1/2}(\widehat{v}_{l-1}(n)) = |\rho_{l-1}(M)|$. This fact and (24) imply that $|v_l(l)| < 1$. In practice, $c(i)$ must be estimated from data. For instance, $c(i)$ can be estimated as

$$\widehat{c}(i) = \sum_{k=-K}^{K} \widehat{C}_{M,x}(0, \ldots, 0, k, k+i) \tag{25}$$

where the integer K must be chosen large enough such that $\widehat{c}(i)$ is approximate to $\sum_{k=-\infty}^{\infty} \widehat{C}_{M,x}(0, \ldots, 0, k, k+i)$.

3. SIMULATION RESULTS

In this section, we provide two simulation examples to demonstrate that the proposed criteria can be used for the design of LPE filters. The first example includes some performance tests to the proposed criteria, while the second example is to employ the designed LPE filter associated with J_4 to deconvolve synthetic seismic data.

Example 1:

The driving input $u(n)$ used was a zero-mean, Exponentially distributed i.i.d. random sequence with variance $\sigma_u^2 = 1$ and $\gamma_3 = 2$. A second-order autoregressive (AR) model $H(z) = 1/A(z)$ with

$$A(z) = 1 + a(1)z^{-1} + a(2)z^{-2} = 1 + 0.7z^{-1} + 0.1z^{-2}$$

was used and $w(n)$ was white Gaussian. The order of the LPE filters to be designed was $p = 2$. The initial conditions $\mathbf{v}_0 = [0, 0]^T$ were used to initialize the previous iterative algorithm associated with $\widetilde{J}$ given by (12) with $K = 10$. The desired LPE filter associated with J_3 was obtained by solving (16) in which $c(i)$ was replaced by $\widehat{c}(i)$ given by (25) with $K = 5$. Note that $F_1 = H(z = 1) = 1/1.8 \ne 0$ (see (2)) for this case (see Theorem 1). For comparison with these HOS based LPE filters, conventional LPE filters were obtained by Burg's algorithm [1,2].

The simulation results are shown in Tables 1 through 3. Observe, from these tables, that when SNR is large (SNR=∞), mean values of all estimated filter coefficients are very close to the true AR parameters. When SNR is low (SNR=5), biases of estimated filter coefficients shown in Tables 2 and 3 are much smaller than those shown in Table 1, and mean square errors (sum of variance and square of bias) of estimated filter coefficients shown in the former are also smaller than those shown in the latter although standard deviations of estimated filter coefficients shown in the latter are smaller than those shown in the former. Therefore, these simulation results indicate that coefficients of the designed HOS based LPE filters by the proposed criteria approximate the true AR parameters well for this case.

Example 2:

The driving input $u(n)$ used was a zero-mean Bernoulli-Gaussian sequence (a sparse spike sequence) with skewness $\gamma_3 = 0$ and kurtosis $\gamma_4 = 0.27$. A third-order nonminimum-phase autoregressive moving average (ARMA) system taken from [8] with transfer function

$$H(z) = \frac{1 + 0.1z^{-1} - 3.2725z^{-2} + 1.41125z^{-3}}{1 - 1.9z^{-1} + 1.1525z^{-2} - 0.1625z^{-3}}$$

was used. The synthetic data $x(n)$ ($N = 512$) shown in Fig. 1(a) were generated based on (1) for SNR = 27 dB and $w(n)$ being white Gaussian. We processed $x(n)$ by the LPE filter of order $p = 40$ associated with J_4 with $K = 300$ in $\widehat{c}(i)$ (see (25)) to get the deconvolved signal $e(n)$ (dotted line) shown in Fig. 1(b). One can see, from Fig. 1(b), that each spike in $u(n)$ (solid line) is associated with a wavelet in $e(n)$ which begins with two opposite peaks and gradually decays due to the remaining phase distortion of the system $H(z)$ (see (F1)). Forty zeros of the LPE filter are shown in Fig. 1(c). One can see, from this figure, that all zeros are inside the unit circle and they scatter uniformly near the unit circle. These results are consistent with the statements described in (F2). The signal $e(n)$ was further processed by Chi-Kung's allpass system deconvolution filter [8] with order equal to 2. The output $\widehat{u}(n)$ (dotted line) of this filter, along with the true input $u(n)$ (solid line), is shown in Fig. 1(d). Note, from Fig. 1(d), that $\widehat{u}(n)$ approximates $u(n)$ well except for a scale factor because the allpass system distortion in $e(n)$ has been considerably removed. These results justify the statements described in (F1).

4. CONCLUSIONS

We have presented two new HOS based criteria $\widetilde{J}$ and J_M given by (4) and (5), respectively, for the design of LPE filters. The designed HOS based LPE filters with measurements corrupted by additive Gaussian noise are identical to the conventional (minimum-phase) LPE filter associated with the case that measurements are noise-free (see Theorem 1). Coefficients of the LPE filter based on J_M can be solved by the computationally efficient Levision-Durbin algorithm but those based on $\widetilde{J}$ must resort to a numerical optimization algorithm. Finally, some simulation results were provided to justify that the proposed HOS based criteria can be used for the design of LPE filters.

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Table 1. Simulation results for the conventional LPE filter obtained by Burg's algorithm

$a(1) = 0.7$, $a(2) = 0.1$, $N = 4096$, 30 independent runs		
	Estimated values (mean±standard deviation)	
SNR	$\widehat{v}(1)$	$\widehat{v}(2)$
∞	0.7023 ± 0.0168	0.1009 ± 0.0164
40	0.6713 ± 0.0173	0.0793 ± 0.0171
10	0.5975 ± 0.0178	0.0312 ± 0.0178
5	0.5262 ± 0.0179	-0.0093 ± 0.0181

Table 2. Simulation results associated with the criterion $\tilde{J}$ with $K = 10$.

$a(1) = 0.7$, $a(2) = 0.1$, $N = 4096$, 30 independent runs		
	Estimated values (mean±standard deviation)	
SNR	$\widehat{v}(1)$	$\widehat{v}(2)$
∞	0.7046 ± 0.0643	0.1010 ± 0.0637
40	0.7108 ± 0.0745	0.1022 ± 0.0705
10	0.7159 ± 0.1128	0.1020 ± 0.0973
5	0.7292 ± 0.1967	0.1162 ± 0.1825

Table 3. Simulation results associated with the criterion J_3 with $K = 5$ in $\widehat{c}(i)$.

$a(1) = 0.7$, $a(2) = 0.1$, $N = 4096$, 30 independent runs		
	Estimated values (mean±standard deviation)	
SNR	$\widehat{v}(1)$	$\widehat{v}(2)$
∞	0.7031 ± 0.0507	0.1003 ± 0.0392
40	0.7049 ± 0.0615	0.0976 ± 0.0494
10	0.7046 ± 0.0886	0.0937 ± 0.0714
5	0.7056 ± 0.1370	0.0923 ± 0.1112

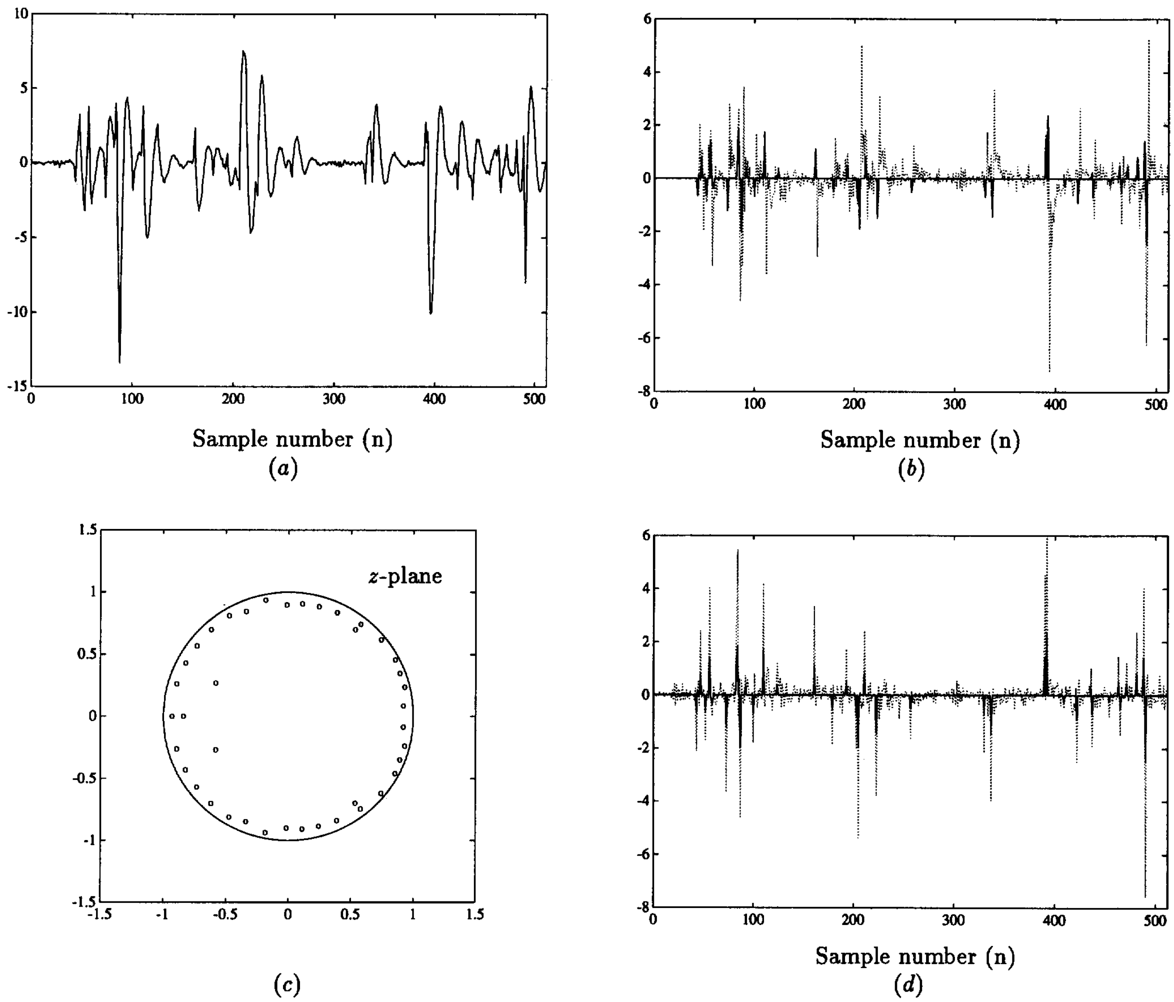

Fig. 1. (a) Synthetic noisy data for $N = 512$ and SNR = 27 dB, (b) the true input signal $u(n)$ (solid line) and the deconvolved signal $e(n)$ (dotted line) obtained by using the 40th-order LPE filter associated with J_4, (c) forty zeros (circles) of the LPE filter, and (d) the true input signal $u(n)$ (solid line) and the output $\widehat{u}(n)$ (dotted line) of Chi-Kung's allpass system deconvolution filter of order equal to 2 with input $e(n)$.